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The use of the state space to record the behavioral effects of subproblems and symmetries in the Tower of Hanoi problem

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This research was designed to focus on the effects of problem *structure* on the *behavior* of subjects solving that problem. The behavior of forty-five adult subjects solving the 4-ring Tower of Hanoi problem was exhibited as paths through the "state space representation" of the problem. Four hypotheses concerning the effects of the problem's structure were tested experimentally. (1) Subjects' paths were both non-random and goal-directed through the problem and its subproblems and the special role of subgoal states was identified. (2) "Episodes" were seen to occur during problem solving corresponding to the consistent solution throughout the problem of subproblems with identical or isomorphic structure, however, (3) the evidence for congruence of subjects' non-minimal solution paths through isomorphic subproblems was inconclusive. (4) The special effect on behavior of symmetries within the structure of the problem was delineated. Directions for further research are outlined.

Introduction

Several studies have recently appeared addressing the relationship between the intrinsic *structure* of a problem and the *strategies* a solver employs or *behavior* evidenced in solving the problem. These include the learning by humans of mathematical structures such as the Klein Group and Cyclic Group of order four (Dienes & Jeeves, 1970; Branca & Kilpatrick, 1972) and the study in humans of "transfer" and "analogy" in related problem solving situations (Reed, Ernst & Banerji, 1974; Thomas, 1974; and Greeno, 1974).

The designers of mechanical problem solvers in Elementary Equation Solving (Bundy, 1975), Geometry Theorem Proving (Goldstein, 1973; Nevins, 1974), and Robot Plan Formation (Sacerdoti, 1973) have also recognized the need for careful analysis of problem structure for efficient problem solving. Perhaps the most celebrated exploitations of problem structure in mechanical problem solving are Gelernter's use of the symmetries within the syntax of a problem's description (1959) and Newell, Shaw & Simon's utilization of a problem's possible subproblem decompositions in their General Problem Solver (Newell *et al.*, 1959; see also Ernst & Newell, 1969).

The research proposed in this paper employs the state space representation of a problem, borrowed from mechanical problem solving theory (Nilsson, 1971), to describe formally a problem's *structure*. The *behaviors* of problem solving subjects are recorded as paths through the state space representation of the problem corresponding to the succession of steps taken or moves made by each subject. It is suggested that features of the problem structure, such as its symmetry and its decomposition into subproblems, might permit the prediction of patterns in subjects' problem solving behavior.

Nilsson has defined the *state space representation* of a problem as the set of distinguishable configurations or situations of the problem together with the set of permitted moves or steps from one problem situation to another. Thus the state space representation of a problem—expressed as a directed graph—consists of an initial state together with all the states that may be reached from the initial state by successive legal moves of the problem. One or more of these successor states are classified as goal states. If a problem's description clearly deliniates the problem's initial state, goal state(s), and its set of legal moves then the state space representation of that problem will be *unique*. Finally, the concept of the state space of a problem can be generalized to the analogous structure for an N-player game, i.e. the game tree or graph.

Besides the work of Nilsson, Banerji (1969), Banerji & Ernst (1972), and others have offered mathematical descriptions to characterize state spaces. This "state space algebra" allows such concepts as problem comparison, decomposition, and extension to be well defined and also allows problem solving studies in the areas of problem analogy, transfer, and generalization to be extremely precise.

In problem solving it may be assumed that the solver operates sequentially upon problem situations (states) to generate successor states, a process which can be described, as noted above, by means of paths through the state space representation of the problem. In this analysis it is not suggested that the problem solver in any way "perceives" the state space as an entity during problem solving. The symmetry properties and subproblem decompositions of the problem are *formal properties of the state space* which may or may not correspond to the geometrical or perceptual properties of the problem readily apparent to the problem solver.

At this stage of research hypotheses have been formulated respecting the paths generated by problem solvers in the state space of a problem (Luger, 1973; Goldin & Luger, 1975). Such hypotheses are (1) motivated by the formal properties of the state space under discussion, and (2) represent the anticipated effects of the problem structure in shaping problem solving behavior. The following hypotheses of a more-or-less general nature are suggested.

Hypothesis 1. Given a subproblem decomposition of a problem, (a) the subject generates non-random, goal-directed paths in the state space representation both of the problem and its subproblems, and (b) when subproblem goal states are entered, the path leaves the state in such a manner as to exit also from the subproblem space.

Hypothesis 2. Identifiable "episodes" occur during problem solving corresponding to the solutions of various subproblems. That is, path segments occur during certain "episodes" which do not constitute the solution of a problem but which do constitute the solution of subproblems of the problem.

Hypothesis 3. The problem solver's paths through subproblems of identical (or isomorphic) structure tend to be congruent.

Hypothesis 4. Given a symmetry within the state space of a problem, there tend to occur in the state space successive path segments congruent *modulo* this symmetry. (Goldin & Luger describe the symmetries of a problem as the group of automorphisms of the problem's state space onto itself.)

While the hypothesis of non-randomness and goal-directedness of subjects' paths through the state space of a problem is perhaps not as interesting as the subsequent hypotheses, these properties are assumed by most accepted theories of problem solving (Simon, 1969; Newell & Simon, 1972). Furthermore, this "path analysis" provides a means for comparing subjects' paths with random paths in order to discern by which local and global variables they differ. The testing of hypothesis 1 also depends on the particular way that the state space of the problem is decomposed since such a decomposition is often not unique. Hypothesis 4 is suggestive of the "insight" phenomenon which "changes the gestalt" of the problem solver and often plays an important role in the eventual problem solution (Wertheimer, 1945). Finally, these hypotheses are not to be regarded as a definitive list but as preliminary and indicative of the kind of analysis possible of effects of problem structure on problem solving behavior.

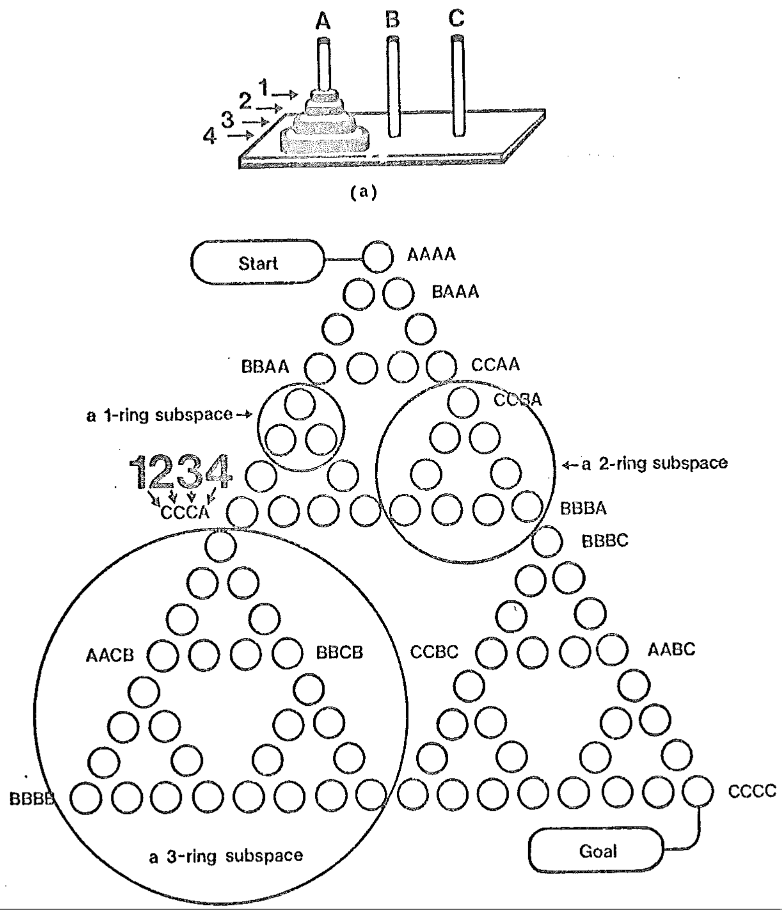
The Tower of Hanoi problem and specific hypotheses for investigation

The preceding ideas are now made more concrete by considering the problem that was used for empirical investigation (Luger, 1973). The Tower of Hanoi problem has been extensively discussed in the literature and its state space considered by Nilsson. It is a natural problem to consider both because its well defined state space has a rich and explicit subproblem structure and because its state space possesses somewhat more symmetry than is immediately apparent in the problem environment.

In the Tower of Hanoi problem four concentric rings (labelled 1, 2, 3, 4 respectively) are placed in order of size, the largest on the bottom, on the first of three pegs (labelled A, B, C). The apparatus is pictured in Fig. 1. The object of the problem is to transfer all the rings from peg A to peg C in the minimum number of moves. Only one ring may be moved at a time, and no larger ring may be placed over a smaller ring on any peg.

Figure 1 also gives the complete state space representation of the Tower of Hanoi problem, each circle standing for a possible position or state of the game. The four letters labelling a state refer to the respective pegs on which the four rings are located. For example, state CCBC means that ring 1, ring 2, and ring 4 are, in their proper order, on peg C, while ring 3 is on peg B. A legal move by the problem solver always effects a transition between states represented by neighboring circles in the state space. The solution path containing the minimum number of moves consists of the fifteen steps from AAAA to CCCC down the right side of the state space diagram.

The Tower of Hanoi has a natural decomposition into nested subproblems. For example, to solve the 4-ring Tower of Hanoi problem, it is necessary at some point to move the largest ring from its original position on peg A to peg C, but before this can be done the three smaller rings must be assembled in their proper order on peg B. The problem of moving the three rings from one peg to another may be termed a 3-ring subproblem, and constitutes a subset of the state space of the 4-ring problem. The 4-ring state space contains three 3-ring subspaces, differing by reason of the position of ring 4. Each subspace becomes a subproblem when one of its entry states is designated as the initial state, and its exit states are designated as goal states. In a like manner, each 3-ring subspace contains three 2-ring subspaces for a total of nine in the 4-ring state space. Each 2-ring subspace may be further decomposed into three 1-ring subspaces, comprising only three states apiece. Note the examples of 1-, 2-, and 3-ring subspaces in Fig. 1. It should be realized that the *structure* of moves within each n-ring subproblem is identical, i.e., even though the "start" peg, the "goal" pegs, and position of the $n+1$ ring differs in each instance, a one-to-one onto mapping exists which preserves the set of possible legal moves in each subproblem. Thus all subproblems of n rings (for any fixed n) are said to be isomorphic.



(b)

FIG. 1. (a) 4-ring Tower of Hanoi board in its initial state. (b) State-space representation of the 4-ring Tower of Hanoi problem. The four letters labelling a state refer to the respective pegs on which the four rings are located. Legal moves effect transitions between adjacent states. Note example of 1-, 2-, and 3-ring subspaces.

Each n-ring subproblem, as well as the main problem, also admits of a symmetry mapping. This automorphism (one-to-one mapping of the space onto itself) maps a goal state of the n-ring problem onto the conjugate goal state which corresponds to transferring the n rings to the other open peg. Were the three pegs of the Tower of Hanoi board to be at the corners of an equilateral triangle the symmetry automorphism would represent the geometric operation of reflection about the altitudes of the triangle.

Four *specific* hypotheses respecting the paths generated by problem solvers in the state space of the Tower of Hanoi problem (and based on the more general hypotheses of the previous section) were formulated and tested. These hypotheses were motivated by the formal properties of the Tower of Hanoi state space and represented anticipated effects of its problem structure in shaping problem solving behavior: (1) (a) the non-randomness and goal-directedness of paths through the state space representation of the Tower of Hanoi problem and its subproblems, and (b) the automatic exit from the subproblem space once the subproblem goal state is achieved, (2) the consistent solution of all isomorphic subproblems on the 1-, 2-, and 3-ring levels via minimal paths as identifiable "episodes" within the total problem solving, (3) the congruence of subject's paths through isomorphic subproblem spaces *prior* to the minimal path solution to subproblems on that level, and (4) the special role of problem symmetry in the problem solving, as evidenced by the interruption of paths and subsequent generation of the "automorphic" image of the interrupted path.

Criteria for fulfillment of these hypotheses were established as follows.

(1) (a) The *non-randomness* of subjects' paths was tested by comparing the number of "corners" or "turns" (as opposed to "straight" sections) in subjects' paths with the number of "corners" or "turns" in paths of the same length generated randomly through the Tower of Hanoi state space. Paths were *goal-directed* when they neither reentered any subproblem once it had been left nor "moved away" from the problem or subproblem's goal state. A measurement of "movement away" from a goal state was possible by establishing a metric on the Tower of Hanoi state space. This metric was the number of states in the shortest path between the subjects' current state and the goal state. (b) The special role of subproblem goal states was investigated by determining the percentage of times that subjects' paths, once having entered the subproblem's goal state, exited from that subgoal in a manner that also exited from the subproblem space. This percentage was compared with the percentage based on random choice of possible exits from the subgoal state (50%).

(2) An *n-ring episode* in solving the Tower of Hanoi problem was defined to occur when a subject executed minimal solutions to 50% or more of all the isomorphs of an n-ring subproblem for a certain period (an $n+1$-ring subproblem) prior to executing minimal solutions to 50% or more of the isomorphs of the $n+1$-ring subproblem. Three such episodes were theoretically possible in solving the 4-ring Tower of Hanoi problem. (See Fig. 2 in which a 2-ring episode is demonstrated.) It was asked whether a significant number of subjects would evidence at least one of these episodes in their solution, and whether any subject would evidence all three.

(3) Congruence of subject's paths through isomorphic subproblem spaces was defined as occurring when any one congruence class of non-minimal solution paths predominated in frequency over the other non-minimal paths. (See Fig. 2.) It was asked whether such congruences would occur for a significant number of subjects on the 2- and 3-ring levels.

(4) The number of subjects was to be established in which an interruption in a path occurred, followed immediately by the symmetric (automorphic) image of the

interrupted path (see Fig. 2, paths 3 and 4). *A priori* it was predicted that such interruptions would occur for half the subjects, since probability dictates that 50% of the subjects' paths would start towards the goal state, and 50% in the symmetrically opposite direction.

It should be evident that this study does not rely solely on conventional statistical tests for establishing the existence of effects in a population of subjects; rather, it suggests some new techniques for establishing the existence of "patterns" in subjects' behavior. For example, it seemed natural to consider "local" properties of the paths: non-random, goal-directed paths were thought to have less "corners" or "turns", less "loops", and to have less "wandering about" than random, undirected paths. The above criteria attempted to make these notions more concrete. Using a metric on the state space, and the analysis of congruence of path segments and the interruptions of paths are simply techniques used to establish the existence of patterns in subjects' behavior.

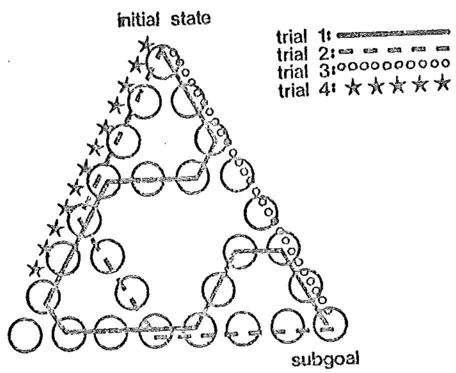


FIG. 2. Analysis of subjects' paths through subproblems. Trial 1: Congruent paths through isomorphic subproblems. All three paths through the 2-ring subproblems are congruent. Trial 2: An "episode" in problem solving. The 2-ring subproblem is consistently solved in the minimum number of steps, while the 3-ring subproblem is not. The state space has been effectively reduced *modulo* its 2-ring subproblems. Trials 3 & 4: Two paths congruent *modulo* a symmetry automorphism of the 3-ring subspace.

Procedures and results

The hypotheses posed above were tested for 58 subjects, 22 male and 36 female. These subjects, college educated adults, were not overly practiced in mathematical games and puzzles. In particular, they had no prior acquaintance with the Tower of Hanoi problem. Each subject was individually interviewed in a well-lighted room, the puzzle placed before them on a desk, and paper and pencil available. The investigator was the only other person present. Once having started the problem, the subject continued to work on it until he or she either gave up or succeeded in moving all the rings from the "begin" to the "goal" state of the problem in the least possible number of moves. The subject could start the problem over at any time or for any reason that he or she wished. The total time spent in solving the problem was usually 15 to 20 minutes. A tape recorder was kept running continuously to record the subjects' verbal responses and the sequence of moves.

The data for seven subjects was discarded for reasons such as the occurrence of interruptions. Six subjects solved the problem in the minimum number of steps on the first attempt. The tape recorded solutions of the remaining 45 subjects were expressed as paths through the state space representation of the Tower of Hanoi problem. Transcripts of the tapes were also prepared to accompany these representations, and both were used to test the proposed hypotheses. The paths representing the solutions of two subjects are given in Fig. 3.

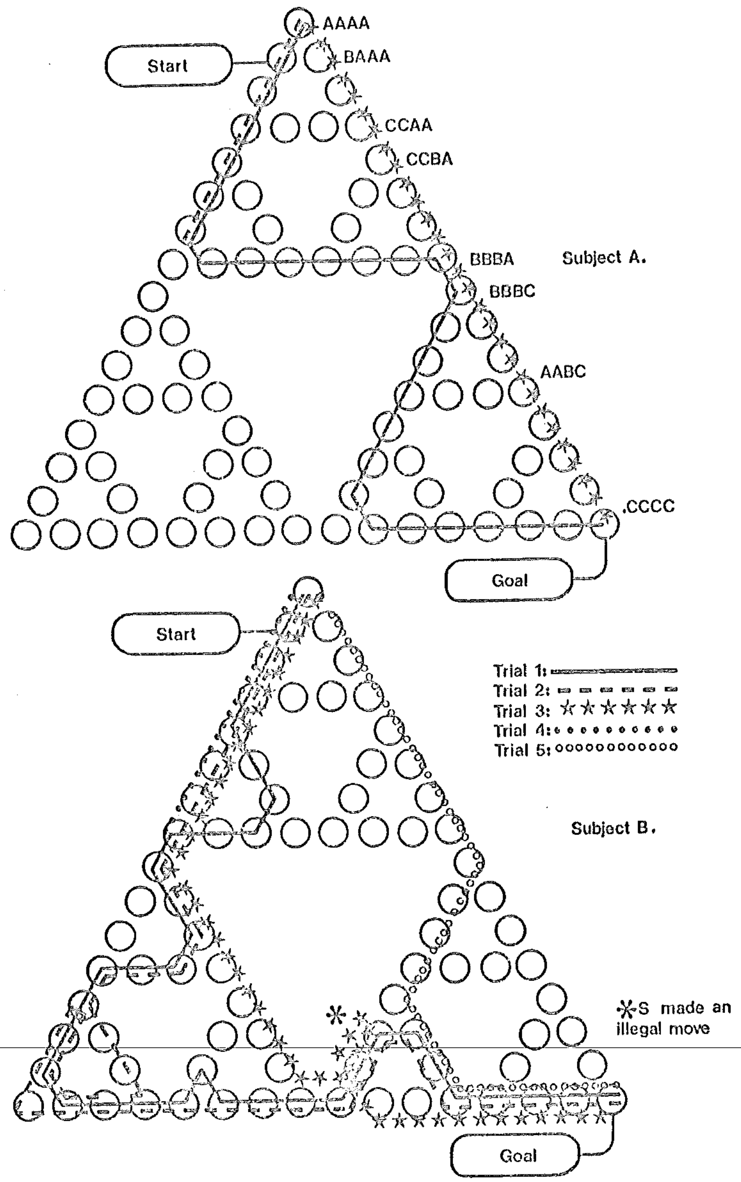


FIG. 3. Paths of two subjects through the state-space of the Tower of Hanoi problem.

HYPOTHESIS 1

(a) In 45 subjects' first trials at solving the problem 95% met the criterion for non-randomness, that is, subjects' first attempt at solving the problem deviated from paths randomly drawn through the Tower of Hanoi state space by more than one standard deviation in the occurrence of "corners" in the paths. Seventy-eight per cent deviated from the random by more than two standard deviations. Of all 131 trials by subjects, 97% met the criterion for non-randomness and 81% deviated from the random by more than two standard deviations. All deviations were in the direction of *fewer* "corners" in the paths. Of 45 subjects' first attempts, 87% satisfied the criterion for goal-directedness; and 93% of the subjects' 133 total attempts satisfied the criterion. Of the 685 paths through 2-ring subproblems, 96% met the criterion for subgoal directedness. Of the 321 paths through 3-ring subproblems, 91% met the criterion.

(b) Of the 685 paths through 2-ring subproblems, 96% met the exit criterion; of the 321 3-ring subproblem paths, 98% met the exit criterion.

In Fig. 3 all paths of subjects A and B deviated from random paths in respect to the number of "corners" by more than 2 standard deviations. (The number of corners per state entered by a random paths was 0·67; S.D. ±10). All paths of subject A were both goal and subgoal directed. For subject B the second 3-ring subproblem of trial 2 was neither goal nor subgoal directed. All paths of subjects A and B exited from a subproblem space once the subgoal state was entered except the fourth 2-ring subproblem and second 3-ring subproblem in the second trial of subject B. (The exit of a random path was 50%).

HYPOTHESIS 2

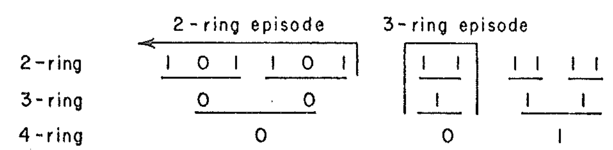
A maximum of three "episodes" were possible, corresponding to minimal solutions of 1-, 2-, and 3-ring subproblems respectively. Of all 51 subjects, 6 (12%) displayed none of these "episodes"; 16 (31%) displayed just one; 22 (43%) displayed exactly two "episodes"; and all three theoretically possible "episodes" were displayed by 7 subjects (14%). As examples of "episodes" consider the two subjects of Fig. 3 as follows.

1 = minimal solution path of a subproblem;

0 = non minimal solution path of a subproblem.

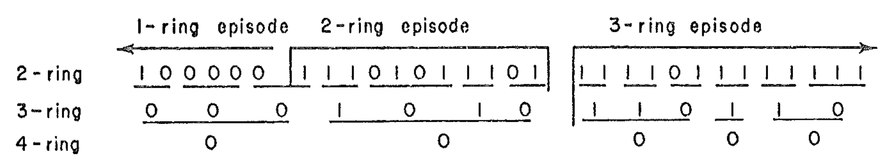
Note that 2-, 3-, and 4-ring (sub)problems are placed over each other in such a manner as to indicate the time sequence (left to right) of problem spaces entered.

The sequence of subproblems entered by subject A:



There was no 1-ring episode. During the 2-ring episode 67% of the 2-ring subproblems were solved minimally and 0% of the 3-ring. During the 3-ring episode 100% of the 3-ring were solved minimally and 0% of the 4-ring.

The sequence of subproblems entered by subject B:



During the 1-ring episode 94% of 1-ring subproblems were solved minimally and only 16% of 2-ring subproblems. During the 2-ring episode 73% of the 2-ring subproblems were solved minimally and only 40% of the 3-ring. During the 3-ring episode 67% of the 3-ring were solved minimally and 0% of the 4-ring.

HYPOTHESIS 3

For the 2-ring subproblem, predominance of one congruence class of non-minimal paths occurred for 7 of 45 subjects (16%); non-predominance occurred for 6 subjects (14%); and 32 subjects (70%) permitted no conclusion to be reached (because of insufficiently many non-minimal paths or an inconclusive distribution of these paths). For the 3-ring subproblem, predominance of one congruence class was shown by 6 subjects (13%); non-predominance by 21 subjects (41%); and 18 subjects (40%) permitted no conclusion.

In trial 1 subject A (Fig. 3) produced congruent non-minimal solution paths through 2-ring subproblems (the second and fifth entered) and also through 3-ring subproblems (the first and second entered). Subject B had a predominance of 2-ring subproblem paths (the second, third, fourth, sixth, eleventh, and fifteenth problems entered) but no predominance on the 3-ring level. In general, due to insufficient numbers of non-minimal solution paths and an inconclusive distribution of these paths, the expected congruence of non-minimal solution paths across isomorphic subproblems was *not* confirmed by the data.

HYPOTHESIS 4

Of 45 subjects, 44% displayed the predicted effect of the problem symmetry, by producing consecutive path segments congruent *modulo* the symmetry transformation of the problem. Seven per cent exhibited this pattern two or more times during the problem solving. This compares reasonably well with the figure of 50%, for whom the phenomena was predicted to occur.

In Fig. 3, Subject A displayed this effect in the second and third trials. Subject B displayed the effect in the fourth and fifth trials.

Summary and conclusions

The author has sought to establish a framework for studying the effects of problem structure on problem solving behavior. The state space representation of a problem is used to describe a problem's structure formally. This formalization allows comparison between problems and subproblems of related structure, and provides a precise description of the symmetry and subproblem decompositions of the problem. The behaviors of problem solving subjects are recorded as paths through the state space according to the steps taken or moves made by the problem solver.

The experimental results of this study seem to confirm an important role played by features of the problem structure in determining patterns in the problem solving behaviors of subjects. In particular, the goal-directed behavior within subproblems and immediate exit from the subproblem space once the subproblem's goal state was achieved, indicate the problem solver's effective "decomposition" of the problem in attempting its solution. The "episodes" within the problem solution indicate the effect on the problem solver of the isomorphic structure of the subproblems. They also seem to indicate, at least in the context of the Tower of Hanoi problem, the problem's solution is found in a "bottom up" progression, with smaller units solved throughout the problem before larger units. This is radically different from the General Problem Solver's mechanical solution to the Tower of Hanoi problem where a "top down" analysis of the differences between START and GOAL states, and a concentration on the largest difference first (i.e. how to get the largest ring to the goal peg) provides a very efficient and parsimonious solution. However, GPS's inability to "recognize" and utilize similarities within a problem's structure, such as human subjects did in the "episodes" of hypothesis 2 and the "symmetries" of hypothesis 4 severely limits its power in wider domains of problem solving (Newell *et al.*, 1959; Ernst & Newell, 1969).

Furthermore, although a 50% or more minimal solution of subproblems was set as an *a priori* criterion in hypothesizing episodes, the actual average percentage of minimal solutions before an episode was much lower than 50% (33% for 1-ring and 23% for 2-ring episodes) and after the episode had begun was substantially above 50% (91% for 1-ring and 87% for 2-ring). This seems to indicate that the subjects' acquisition of an *n*-ring structure (by producing a minimal path wherever this structure was encountered throughout the problem) was in an "all or nothing" fashion. This analysis is trivially true of 3-ring episodes since problem solving ceased at the first minimal solution of the 4-ring problem.

Finally, the symmetry structure within the problem was reflected in the problem solvers' interrupted paths. These interruptions often culminated in the solution of the problem.

The author defines the *structure* of a problem to be the formal properties of its state space and lets the term *strategy* refer to particular rules or procedures for taking steps within the state space. Different individuals may employ different strategies in solving the same problem, and the same individual may employ different strategies in solving different but structurally related problems. The present paper does not explain strategies *per se* but hypothesizes that even in the context of different strategies, certain patterns of behavior tend to occur as a consequence of the structure of the problem.

Further, it may be speculated that a subject's *cognitive structures* ought to be defined to include the symmetry detections and subproblem decompositions that the subject can employ during problem solving. These determine the states of the environment that the subject treats as distinct and those he or she is able to treat as equivalent for solving the problem. This ability of the problem solving subject to treat different states of the environment as *functionally equivalent* corresponds to an effective "reduction" of the state space and solution of the problem (Goldin & Luger, 1975).

Finally, although the suggested general hypotheses of the introductory section, were only tested for a single problem and with a limited population of subjects, current research by the author is testing these hypotheses in several other specific problem solving situations. These include a problem whose structure is isomorphic to the Tower of Hanoi (Luger, 1976), and the analysis of "transfer" effects on subjects solving problems of related structure (Bauer & Luger, 1976).

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